ON THE STABILITY OF ROTATION OF A RIGID BODY SUSPENDED ON A STRING AND POSSESSING AN ELLIPSOIDAL CAVITY COMPLETELY FILLED WITH AN IDEAL INCOMPRESSIBLE FLUID

(OB USTOICHIVOSTI VRASHCHENIIA NA STRUNE TVERDOGO TELA S ELLIPSOIDAL'NOI POLOST'IU, TSELIKOM NAPOLNENNOI IDEAL'NOI NESZHIMAEMOI ZHIDKOST'IU)

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The present study of the stability of the rotation on a string of a solid with a cavity filled with an ideal fluid, again uses the method developed in [1]. The principle of this method consists in the description of the motion of the fluid with respect to a system of coordinates fixed with respect to the rigid body, performing motion under the action of the fluid, its own weight and reaction forces of the suspension.

The problem considered below was, with the fluid absent, investigated in [2]. Another limiting case corresponding to a string of zero length is the well known problem of Sobolev, investigated in [1 and 3]. We shall utilise some of the methods from the above works.

1. Let a symmetric rigid body suspended on an ideally flexible, inextensible and inertialess string of length l, rotate with a constant angular velocity ω about a vertical axis in a steady motion (Fig. 1). Inside the body, there is a cavity, which has the form of an ellipsoid of revolution, and which is completely filled with an ideal incompressible fluid. The axes of symmetry of the cavity and of the body coincide. During the steady motion the fluid rotates with the body, as if they were a single rigid body. We shall investigate the stability of such a stationary motion.

2. We shall consider first the derivation of the differential equations of the body in question, assuming that this motion does not differ much from the above mentioned stationary rotation whose angular velocity about a vertical axis is ω.

We shall introduce the fixed system of coordinates $\xi\eta\zeta$ which has the vertical axis ζ

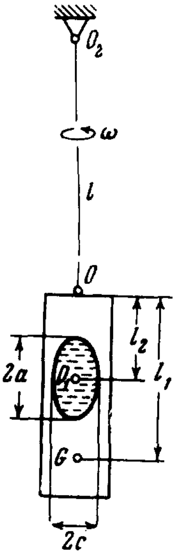

FIG. 1

and its origin O_2 at the point of attachment of the string to a fixed support, and also the moving system $\xi^\circ\eta^\circ\zeta^\circ$ with its origin O_2 at the point at which the body is tied to the string (Fig. 2).

The corresponding axes of these systems are parallel. At the point O we shall also locate the origin of the system of coordinates xyz rigidly connected to the body. Axis z of this system will coincide with the axes of symmetry of the body and the cavity, and the axes x and y will lie in the plane perpendicular to the axis z so as to form together a rectangular system of coordinates.

The position of the string with respect to the system $\xi\eta\zeta$ will be defined by two angles λ and μ; at the same time, λ is the angle between the axis ζ and the projection on the axis $\eta\zeta$ of the straight line directed upwards along the string, while μ is the angle between that line and the plane $\eta\zeta$ (Fig. 2).

The position of the solid body with respect to the system $\xi^\circ\eta^\circ\zeta^\circ$ is determined by three Euler-Krylov angles (Fig. 3):

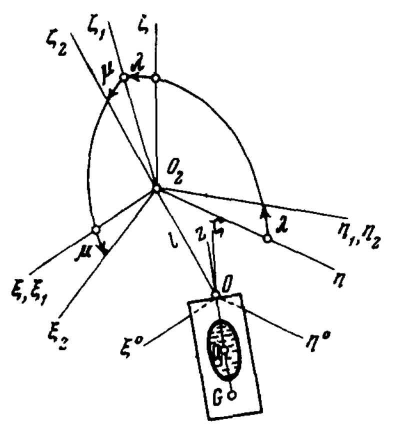
FIG. 2

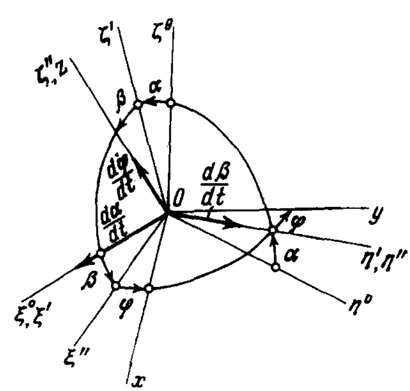
FIG. 3

the angle α between the axis ζ° and the projection of the axis of symmetry of the body (axis z) on the plane $\eta^\circ\zeta^\circ$; the angle β between the axis z and the same plane $\eta^\circ\zeta^\circ$ and the angle φ between the axis x and the auxiliary axis ξ'' which is at the right angles to the z-axis and lies in the plane $\xi^\circ z$.

The cosine table of the angles between the systems of coordinates $\xi\eta\zeta$ and xyz or (which is the same), between the $\xi^\circ\eta^\circ\zeta^\circ$ and xyz in the above case have the form

	ξ	η	ζ	
x	$\cos\beta\cos\varphi$	$\sin\alpha\sin\beta\cos\varphi + \cos\alpha\sin\varphi$	$-\sin\beta\cos\alpha\cos\varphi + \sin\alpha\sin\varphi$	(2.1)

y	$-\cos\beta\sin\varphi$	$-\sin\alpha\sin\beta\sin\varphi + \cos\alpha\cos\varphi$	$\sin\beta\cos\alpha\sin\varphi + \sin\alpha\cos\varphi$
z	$\sin\beta$	$-\sin\alpha\cos\beta$	$\cos\alpha\cos\beta$

It is easy to see from Fig. 3 that the projections ω_x, ω_y and ω_z of the angular velocity of the system of coordinates xyz on its own axes will be

$$\omega_x = \frac{d\alpha}{dt}\cos\beta\cos\varphi + \frac{d\beta}{dt}\sin\varphi$$
$$\omega_y = -\frac{d\alpha}{dt}\cos\beta\sin\varphi + \frac{d\beta}{dt}\cos\varphi \tag{2.2}$$
$$\omega_z = \frac{d\alpha}{dt}\sin\beta + \frac{d\varphi}{dt}$$

For future use we shall also define the coordinates of the point O at which the body is attached to the string in terms of the system $\xi\eta\zeta$. We have

$$\xi_0 = -l\sin\mu, \qquad \eta_0 = l\cos\mu\sin\lambda, \qquad \zeta_0 = -l\cos\mu\cos\lambda \tag{2.3}$$

The expression for the kinetic energy of the solid can be written in the form

$$T = \frac{1}{2}mv_0^2 + m\begin{vmatrix}\omega_x & \omega_y & \omega_z\\ x_G & y_G & z_G\\ (v_0)_x & (v_0)_y & (v_0)_z\end{vmatrix} + \frac{1}{2}[(A+ml_1^2)(\omega_x^2+\omega_y^2)+C\omega_z^2] \tag{2.4}$$

Here m is the mass of the rigid body; $A + ml_1^2 = B + ml_1^2$ and C are its moments of inertia with respect to the axes x, y and z; v_0 is the absolute velocity of the origin O of the coordinates; x_G, y_G, z_G are the coordinates of the center of gravity of the body in the system xyz; l_1 is the distance between the center of gravity of the body and the point at which it is attached to the string (Fig. 1).

From the expression (2.4) and the formulas (2.2), and (2.3), we obtain an explicit expression for the kinetic energy in terms of generalized coordinates α, β, λ, and μ and generalized velocities $d\alpha/dt$, $d\beta/dt$, $d\lambda/dt$, $d\mu/dt$, and $d\varphi/dt$ (the generalized coordinate φ is cyclic), which is

$$\begin{aligned}T = {} & \frac{1}{2}m\left[l^2\left(\frac{d\mu}{dt}\right)^2 + l^2\cos^2\mu\left(\frac{d\lambda}{dt}\right)^2\right] + ml_1l\frac{d\alpha}{dt}\cos\beta\left[\frac{d\mu}{dt}\sin\mu\sin(\alpha-\lambda) + \right.\\ & \left. + \frac{d\lambda}{dt}\cos\mu\cos(\alpha-\lambda)\right] + ml_1l\frac{d\beta}{dt}\left[\frac{d\mu}{dt}\cos\beta\cos\mu + \right.\\ & \left. + \frac{d\mu}{dt}\sin\mu\sin\beta\cos(\alpha-\lambda) - \frac{d\lambda}{dt}\cos\mu\sin\beta\sin(\alpha-\lambda)\right] + \\ & + \frac{1}{2}(A+ml_1^2)\left[\left(\frac{d\alpha}{dt}\right)^2\cos^2\beta + \left(\frac{d\beta}{dt}\right)^2\right] + \frac{1}{2}C\left[\frac{d\alpha}{dt}\sin\beta + \frac{d\varphi}{dt}\right]^2\end{aligned} \tag{2.5}$$

We shall now express the equations of motion of the solid by using the second Lagrange's method. Restricting ourselves to the infinitesimals of the first order with respect to the coordinates λ, μ, α, and β and their time derivatives, we obtain the following set of linear differential equations with constant coefficients:

$$ml\left(l\,\frac{d^2\lambda}{dt^2}+l_1\,\frac{d^2\alpha}{dt^2}\right)=Q_\lambda,\qquad ml\left(l\,\frac{d^2\mu}{dt^2}+l_1\,\frac{d^2\beta}{dt^2}\right)=Q_\mu$$

$$A\,\frac{d^2\alpha}{dt^2}+C\beta\,\frac{d^2\varphi}{dt^2}+C\,\frac{d\varphi}{dt}\,\frac{d\beta}{dt}+ml_1\left(l\,\frac{d^2\lambda}{dt^2}+l_1\,\frac{d^2\alpha}{dt^2}\right)=Q_\alpha \tag{2.6}$$

$$A\,\frac{d^2\beta}{dt^2}-C\,\frac{d\alpha}{dt}\,\frac{d\varphi}{dt}+ml_1\left(l\,\frac{d^2\mu}{dt^2}+l_1\,\frac{d^2\beta}{dt^2}\right)=Q_\beta$$

$$C\,\frac{d}{dt}\,\frac{d\varphi}{dt}=Q_\varphi$$

3. The right hand sides of the equations (2.6) represent generalized forces which correspond to selected independent coordinates of the rigid body. These quantities appear as coefficients on perturbing the corresponding generalized coordinates appearing in the expression for the elementary work δW done by active forces acting on the b ody during its arbitrary displacement. We have [4]

$$\delta W=\mathbf{P}\delta\mathbf{r}_0+\mathbf{L}_0\delta\boldsymbol{\gamma} \tag{3.1}$$

where $\mathbf{P}$ is the principal vector of all the active forces applied to the body and $\mathbf{L}_0$ is their principal moment with respect to the point O; $\delta\mathbf{r}_0$ is a possible displacement of the point O, $\delta\gamma$ is the vector of possible angular displacement of the body.

We shall denote by P_ξ, P_η, and P_ζ, the projections of the vector $\mathbf{P}$ on the axes ξ, η and ζ and by L_α, L_β, and L_φ the sums of the moments of the active forces with respect to the axes ξ°, η' and z (Fig. 3).

Then, for small angles α and β we shall obtain the following expression for δW

$$\delta W=P_\xi\delta\xi_0+P_\eta\delta\eta_0+P_\zeta\delta\zeta_0+L_\alpha\delta\alpha+L_\beta\delta\beta+L_\varphi\delta\varphi \tag{3.2}$$

or, using (2.3) and assuming that the angles λ and μ are also small quantities

$$\delta W=l\,(P_\eta+\lambda P_\zeta)\,\delta\lambda+l\,(P_\zeta\mu-P_\xi)\,\delta\mu+L_\alpha\delta\alpha+L_\beta\delta\beta+L_\varphi\delta\varphi \tag{3.3}$$

From the last expression it follows

$$Q_\lambda=l\,(P_\eta+\lambda P_\zeta),\;\;Q_\mu=l\,(P_\zeta\mu-P_\xi),\;\;Q_\alpha=L_\alpha,\;\;Q_\beta=L_\beta,\;\;Q_\varphi=L_\varphi \tag{3.4}$$

The active forces acting on the body are the force of gravity mg and the forces exerted on the body by the enclosed fluid. From this it follows that the generalized force Q_φ is equal to zero. In fact, it represents the sum of the moments of the force of gravity and of the forces due to the pressure of the fluid with respect to the z-axis. However, because of the symmetry, the center of gravity of the body is on the z-axis and the vector representing the pressure of the fluid on the body crosses this axis everywhere.

Considering the cosine table (2.1), the formulas (3.3) and the smallness of the angles α, β, λ and μ, the equations of motion (2.6) can be written in the form

$$ml\left(l\,\frac{d^2\lambda}{dt^2}+l_1\,\frac{d^2\alpha}{dt^2}\right)=-mgl\lambda+l\,[F_x\sin\varphi+F_y\cos\varphi-F_z\alpha]+lF_z\lambda \tag{3.5}$$

$$ml\left(l\,\frac{d^2\mu}{dt^2}+l_1\,\frac{d^2\beta}{dt^2}\right)=-\,mgl\mu-l\,[F_x\cos\varphi-F_y\sin\varphi+F_z\beta]+lF_z\mu$$

$$(A+ml_1^2)\frac{d^2\alpha}{dt^2}+ml_1l\,\frac{d^2\lambda}{dt^2}+C\,\frac{d\varphi}{dt}\,\frac{d\beta}{dt}=-\,mgl_1\alpha+M_x\cos\varphi-M_y\sin\varphi$$

$$(A+ml_1^2)\frac{d^2\beta}{dt^2}+ml_1l\,\frac{d^2\mu}{dt^2}-C\frac{d\varphi}{dt}\,\frac{d\alpha}{dt}=-mgl_1\beta+M_x\sin\varphi+M_y\cos\varphi$$

$$C\,\frac{d}{dt}\,\frac{d\varphi}{dt}=0$$

In these equations, besides the quantities already known, g is the acceleration due to gravity and

$$F_x=\iint_\sigma p^*\cos x\nu\,d\sigma,\qquad F_y=\iint_\sigma p^*\cos y\nu\,d\sigma,\qquad F_z=\iint_\sigma p^*\cos z\nu\,d\sigma \tag{3.6}$$

are the projections on the axes x, y, and z of the principal vector of the pressure forces exerted by the fluid on the body.

Similarly

$$M_x=\iint_\sigma (p^*y\cos z\nu-p^*z\cos y\nu)\,d\sigma,\qquad M_y=\iint_\sigma (p^*z\cos x\nu-p^*x\cos z\nu)\,d\sigma \tag{3.7}$$

are the projections on the axes x and y of the principal moment of the forces mentioned earlier, with respect to the point of suspension O. The integration in the relations (3.6) and (3.7) is made along the side of the cavity σ; $\cos x\nu$, $\cos y\nu$ and $\cos z\nu$ are the direction cosines of the external normal ν to the surface of the cavity; x, y and z are the running coordinates of the element of surface $d\sigma$.

The formulas (3.6) and (3.7) see [5]) can be replaced by the following

$$F_x=\iiint_\tau \frac{\partial p^*}{\partial x}\,d\tau,\qquad F_y=\iiint_\tau \frac{\partial p^*}{\partial y}\,d\tau;\qquad F_z=\iiint_\tau \frac{\partial p^*}{\partial z}\,d\tau \tag{3.8}$$

$$M_x=\iiint_\tau\left(y\,\frac{\partial p^*}{\partial z}-z\,\frac{\partial p^*}{\partial y}\right)d\tau,\qquad M_y=\iiint_\tau\left(z\,\frac{\partial p^*}{\partial x}-x\,\frac{\partial p^*}{\partial z}\right)d\tau \tag{3.9}$$

Here, the integration is performed over the whole volume of the cavity.

4. In order to determine the pressure p^* inside the fluid in terms of the coordinates x, y and z and the time t, we shall consider the equations of motion of the fluid with respect to the moving system of coordinates xyz stationary with respect to the rigid body. In those equations we shall consider the projections u_x, u_y, and u_z of the relative velocity of any particle of the fluid, and their derivatives with respect to the coordinates, to be small. Disregading, furthermore, the products of small quantities, we get

$$\frac{\partial u_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} - w_x^{\,e} - w_x^{\,c} - g\cos\zeta x \qquad (xyz) \qquad (4.1)$$

The symbol (xyz) means that the two other formulas are obtained by circular permutations; w_x^e, w_y^e, and w_z^e are the projections on the x, y and z axes of the translational acceleration w_x^c, w_y^c, and w_z^c of the Coriolis acceleration of a fluid particle, $\cos\zeta x$, $\cos\zeta y$, and $\cos\zeta z$ are the direction cosines of the axis ζ in the xyz system of coordinates. The latter terms are equal to the quantities $\alpha\sin\varphi - \beta\cos\varphi$, $\alpha\cos\varphi + \beta\sin\varphi$ and 1 respectively (Fig. 3 and (2.1)), with the accuracy of up to the second order terms of the small angles α and β. The projections of translational acceleration are given by the formulas

$$w_x^{\,e} = w_x^{\,0} + \frac{d\omega_y}{dt}z - \frac{d\omega_z}{dt}y + \omega_x(x\omega_x + y\omega_y + z\omega_z) - \omega^2 x \qquad (xyz) \qquad (4.2)$$

Here w_x^0, w_y^0, and w_z^0 are the projections on the axes x, y and z of the total acceleration of the point O, which is the origin of xyz coordinate system.

These projections can be expressed with accuracy of up to the second order in terms of the derivatives of the angles α, β, λ and μ by

$$w_x^{\,0} = -l\,\frac{d^2\mu}{dt^2}\cos\varphi + l\,\frac{d^2\lambda}{dt^2}\sin\varphi, \qquad w_y^{\,0} = l\,\frac{d^2\mu}{dt^2}\sin\varphi + l\,\frac{d^2\lambda}{dt^2}\cos\varphi, \qquad w_z^{\,0} = 0 \qquad (4.3)$$

With the same accuracy we obtain by (2.2),

$$\omega_x = \frac{d\alpha}{dt}\cos\varphi + \frac{d\beta}{dt}\sin\varphi, \qquad \omega_y = -\frac{d\alpha}{dt}\sin\varphi + \frac{d\beta}{dt}\cos\varphi, \qquad \omega_z = \frac{d\varphi}{dt} = \omega \qquad (4.4)$$

From the last equation of (3.5) it follows that $\omega = \text{const}$, and if we disregard an unimportant constant, then $\varphi = \omega t$. Neglecting furthermore in the expressions (4.2) the terms of the second order in ω_x and ω_y (which are of the order of $d\alpha/dt$ and $d\beta/dt$), and also the terms which include the derivative of ω_z with respect to time, we get

$$\begin{aligned} w_x^{\,e} &= -l\,\frac{d^2\mu}{dt^2}\cos\varphi + l\,\frac{d^2\lambda}{dt^2}\sin\varphi + \frac{d\omega_y}{dt}z + \omega\omega_x z - \omega^2 x \\ w_y^{\,e} &= l\,\frac{d^2\mu}{dt^2}\sin\varphi + l\,\frac{d^2\lambda}{dt^2}\cos\varphi - \frac{d\omega_x}{dt}z + \omega\omega_y z - \omega^2 y \\ w_z^{\,e} &= \frac{d\omega_x}{dt}y - \frac{d\omega_y}{dt}x + \omega\omega_x x + \omega\omega_y y \end{aligned} \qquad (4.5)$$

The projections of the Coriolis acceleration on the x, y, and z axes are equal [4] to

$$w_x^{\,c} = 2(\omega_y u_z - \omega_z u_y), \qquad w_y^{\,c} = 2(\omega_z u_x - \omega_x u_z), \qquad w_z^{\,c} = 2(\omega_x u_y - \omega_y u_x) \qquad (4.6)$$

Here also, one should neglect the terms in which ω_x and ω_y, appear as multipliers and write $\omega_z = \omega$.

Taking the formulas (4.2) and (4.6) into account, the equations (4.1) take the form

$$\frac{\partial u_x}{\partial t} - 2\omega u_y = -\frac{1}{\rho}\frac{\partial p_1}{\partial x} - 2z\frac{d\omega_y}{dt}$$
$$\frac{\partial u_y}{\partial t} + 2\omega u_x = -\frac{1}{\rho}\frac{\partial p_1}{\partial y} + 2z\frac{d\omega_x}{dt}, \quad \frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p_1}{\partial z} \tag{4.7}$$

Where

(4.8)

$$p_1 = p^* + \rho\left[\omega z(x\omega_x + y\omega_y) - {}^1/_2\omega^2(x^2 + y^2)\right] + \rho g\left[x(\alpha\sin\varphi - \beta\cos\varphi) + \right.$$
$$\left. + y(\alpha\cos\varphi + \beta\sin\varphi) + z\right] + \rho(xw_x^0 + yw_y^0 + zw_z^0) + \rho z\left(y\frac{d\omega_x}{dt} - x\frac{d\omega_y}{dt}\right)$$

and ω_x, ω_y, w_x^0, w_y^0, and w_z^0 are determined by the equations (4.3) and (4.4).

The functions u_x, u_y, and u_z must, furthermore, satisfy the condition of incompressibility

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{4.9}$$

and the boundary condition

$$u_x \cos x\nu + u_y \cos y\nu + u_z \cos z\nu = 0 \tag{4.10}$$

The latter equation means that the projection of the relative velocity of a fluid particle in contact with the boundary of the cavity, onto the axis normal to this boundary, is equal to zero.

5. Following the work of Sobolev [3], we search for u_x, u_y and u_z in the form

(5.1)

$$u_x = U_x(t)(z + l_2), \qquad u_y = U_y(t)(z + l_2), \qquad u_z = -\frac{c^2}{a^2}\left[xU_x(t) + yU_y(t)\right]$$

Here $U_x(t)$ and $U_y(t)$ are determinable functions of time*; l_2 is the distance between the center of the ellipsoidal cavity and the origin of the xyz coordinate system (Fig. 1).

We denote the major and the minor semi-axe of the ellipsoidal cavity by a and c respectively. Then, its equation will take the form

$$\frac{x^2 + y^2}{a^2} + \frac{(z + l_2)^2}{c^2} = 1 \tag{5.2}$$

and the direction cosines of the normal ν in the xyz coordinate system will be

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* Also see the note at the beginning of the work [1].

$$\cos x\nu = \frac{xc^2}{D}, \qquad \cos y\nu = \frac{yc^2}{D}, \cos z\nu = \frac{a^2(z+l_2)}{D} \tag{5.3}$$

$$D = \sqrt{c^4(x^2+y^2)+a^4(z+l_2)^2} \tag{5.4}$$

By direct substitution of the relations (5.1) and (5.3) into the boundary condition (4.10) and using the condition of incompressibility (4.9) we can see that the latter is identically satisfied.

Let us now consider the system (4.7). Multiplying its first equation by cos $x\nu$, the second by cos $y\nu$ and the third by cos $z\nu$ and adding, we obtain by utilising also (5.1) and (5.3).

$$\frac{1}{\rho}\frac{\partial p_1}{\partial n} = \frac{2c^2}{D} z \left\{ x\left[\omega U_y(t) - \frac{d\omega_y}{dt}\right] + y\left[-\omega U_x(t) + \frac{d\omega_x}{dt}\right]\right\} + \\ + \frac{2c^2 l_2\omega}{D}\left[xU_y(t) - yU_x(t)\right] \tag{5.5}$$

The last relation becomes an identity if we write

$$p_1 = \rho z\, [xP^*(t) + yQ^*(t)] + \rho\, [xP_1^*(t) + yQ_1^*(t)] \tag{5.6}$$

and choose the functions $P^*(t)$, $P_1^*(t)$, $Q^*(t)$ and $Q_1^*(t)$ as follows:

$$P^*(t) = \frac{2c^2}{a^2+c^2}\left[\omega U_y(t) - \frac{d\omega_y}{dt}\right], \qquad P_1^*(t) = \frac{2l_2}{a^2+c^2}\left[a^2\frac{d\omega_y}{dt} + \omega U_y(t)c^2\right]$$
$$Q^*(t) = \frac{2c^2}{a^2+c^2}\left[-\omega U_x(t) + \frac{d\omega_x}{dt}\right], \qquad Q_1^*(t) = -\frac{2l_2}{a^2+c^2}\left[a^2\frac{d\omega_x}{dt} + \omega U_x(t)c^2\right] \tag{5.7}$$

Let us now multiply the second equation of the system (4.7) by $i = \sqrt{-1}$ and add it to the first. Then by using the equalities (5.1), (5.6), (5.7) and simplyfing by the factor $z + l_2$ we obtain the relation necessary for the determination of the functions $U_x(t)$ and $U_y(t)$, i.e.

$$\left[(a^2+c^2)\frac{d}{dt} + 2\omega a^2 i\right][U_x(t) + iU_y(t)] = 2a^2 i\left(\frac{d\omega_x}{dt} + i\frac{d\omega_y}{dt}\right) \tag{5.8}$$

6. Now, let us investigate the system (3.5). Let us multiply its second and fourth equations by $i = \sqrt{-1}$ and add the first to the second and the third to the fourth. Then, let us also introduce the complex functions of the real variable t

$$\zeta^* = \alpha + i\beta, \qquad z^* = \lambda + i\mu \tag{6.1}$$

Taking into consideration the third equality of (4.4) we have

$$(A + ml_1^2)\frac{d^2\zeta^*}{dt^2} - iC\omega\frac{d\zeta^*}{dt} + ml_1 l\frac{d^2z^*}{dt^2} + mgl_1\zeta^* = (M_x + iM_y)e^{i\omega t}$$
$$m\left(l\frac{d^2z^*}{dt^2} + l_1\frac{d^2\zeta^*}{dt^2}\right) + (mg - F_z)z^* = -i(F_x + iF_y)e^{i\omega t} - F_z\zeta^* \tag{6.2}$$

To calculate the expressions $F_x + iF_y$, F_z, and $M_x + iM_y$, which appear in the right hand sides of the system (6.2), we substitute into the formulas (3.8) and (3.9) the expressions for the pressure p^* found by eliminating the variable p_1 between the relations (4.8) and (5.6). Also we note, that

$$\iiint_\tau \rho xz\, d\tau = \iiint_\tau \rho xy\, d\tau = \iiint_\tau \rho zy\, d\tau = 0, \qquad \iiint_\tau \rho\,(x^2 - y^2)\, d\tau = 0$$
$$\iiint_\tau \rho x\, d\tau = \iiint_\tau \rho y\; d\tau = 0, \qquad \iiint_\tau \rho\, d\tau = m_1, \qquad \iiint_\tau \rho z\, d\tau = -\, m_1 l_2 \quad (6.3)$$
$$\iiint_\tau \rho\,(z^2 - y^2)\, d\tau = \iiint_\tau \rho\,(z^2 - x^2)\, d\tau = \frac{4}{15}\,\pi\rho a^2 c\,(c^2 - a^2) + m_1 l_2^2$$

Here, in addition to the symbols already seen earlier, ρ and m_1 represent respectively the specific density and the mass of the fluid which fills the ellipsoidal cavity of the rigid body.

If, furthermore, we notice that according to the formula (4.3), (4.4) and (6.1)

$$\omega_x + i\omega_y = \frac{d\zeta^*}{dt}\, e^{-i\omega t}, \qquad w_x^0 + iw_y^0 = il\,\frac{d^2 z^*}{dt^2}\, e^{-i\omega t} \quad (6.4)$$

then, after a few rather simple calculations and the use of the equalities (5.7), (5.8) we get the required expressions of $F_x + iF_y$ and F_z

$$F_x + iF_y = -\, m_1 i\left[l\,\frac{d^2 z^*}{dt^2} + l_2\,\frac{d^2\zeta^*}{dt^2} + g\zeta^*\right] e^{-i\omega t}, \qquad F_z = -\, m_1 g \quad (6.5)$$

and also the differential equation for $M_x + iM_y$, which is

$$M_x + iM_y = k\,\frac{-(c^2 - a^2)\, d^3\zeta^*/dt^3 + 2\omega i c^2\, d^2\zeta^*/dt^2 + (a^2 + c^2)\,\omega^2\, d\zeta^*/dt}{(a^2 + c^2)\, d/dt + 2\omega a^2 i}\, e^{-i\omega t} -$$
$$-\left[m_1 l_2^2\,\frac{d^2\zeta^*}{dt^2} + m_1 g l_2 \zeta^* + m_1 l_2 l\,\frac{d^2 z^*}{dt^2} + i\omega k\,\frac{d\zeta^*}{dt}\right] e^{-i\omega t} \quad (6.6)$$

where

$$k = {}^4/_{15}\pi\rho a^2 c\,(c^2 - a^2) \quad (6.7)$$

7. Eliminating $F_x + iF_y$, F_z and $M_x + iM_y$ from (6.2) by using the relation (6.5) and (6.6) we obtain the following system of equations with respect to the complex functions of time ζ^* and z^*

$$(A^* + k\eta)\,\frac{d^3\zeta^*}{dt^3} - i\omega\,[C + (A^* + k)\eta]\,\frac{d^2\zeta^*}{dt^2} + [K - \omega^2\,(C\eta - k\eta + k)]\,\frac{d\zeta^*}{dt} -$$
$$-\, i\omega\eta K\zeta^* + (ml_1 + m_1 l_2)\, l\,\frac{d^3 z^*}{dt^3} - i\omega\eta l\,(ml_1 + m_1 l_2)\,\frac{d^2 z^*}{dt^2} = 0 \quad (7.1)$$
$$l\,\frac{d^2 z^*}{dt^2} + z_c\,\frac{d^2\zeta^*}{dt^2} + gz^* = 0$$

Where

$$A^* = A + ml_1^2 + m_1 l_2^2, \qquad \eta = \frac{c^2 - a^2}{c^2 + a^2}, \qquad z_c = \frac{ml_1 + m_1 l_2}{m + m_1} \qquad (7.2)$$
$$K = g(ml_1 + m_1 l_2),$$

We assume that the solution of (7.1) has the form

$$\zeta^* = \zeta^0 e^{i\lambda t}, \qquad z^* = z^0 e^{i\lambda t} \qquad (7.3)$$

At the same time, the characteristic equation reduces to the following:

$$f(\lambda; \omega) = (A^0 + k\eta)\lambda^5 - \omega\,[C + (A^0 + k)\eta]\lambda^4 + [-K - (A^* + k\eta)\tfrac{g}{l} + C\omega^2\eta + k\omega^2(1 - \eta)]\lambda^3 + [K\eta + C\tfrac{g}{l} + \tfrac{g}{l}(A^* + k)\eta]\,\omega\lambda^2 + \tfrac{g}{l}[K - k\omega^2(1 - \eta) - C\omega^2\eta]\,\lambda - K\omega\eta\tfrac{g}{l} = 0 \qquad (7.4)$$

Where

$$A^0 = A^* - z_c(ml_1 + m_1 l_2) \qquad (7.5)$$

The equations (7.1) together with the equality $\omega_z = \omega = \text{const}$ describe the motion of the body in question. Hence the investigation of the stability of its motion reduces to the study of behaviour of the functions (7.3). Obviously, the condition that the roots of the characteristic equation (7.4) are real, is the criterion of the stability of motion. Below, we shall try to determine this condition.

8. First, we shall consider a few particular cases. *a*) The length of the string $l \to 0$. Then, the characteristic equation (7.4) takes the form

$$(A^* + k\eta)\lambda^3 - [C + (A^* + k)\eta]\,\omega\lambda^2 - [K - k\omega^2(1 - \eta) - C\omega^2\eta]\,\lambda + K\omega\eta = 0 \qquad (8.1)$$

Note, that even as far as the notations are concerned, the equation (8.1) coincides with the equation obtained in papers [1] and [3] in the investigation of the stability of a gyroscope with an ellipsoidal cavity completely filled with an ideal fluid.* *b*) The cavity has a spherical form. Then $a = c$ and, according to equation (6.7) and the third relation of (7.2), $k = 0$, $\eta = 0$. The characteristic equation (7.4) takes the form

$$\lambda\left\{\lambda^4 - \frac{C}{A^0}\,\omega\lambda^3 - \frac{g}{l}\left[1 + \frac{(ml_1 + m_1 l_2)(l + z_c)}{A^0}\right]\lambda^2 + \frac{C}{A^0}\,\omega\,\frac{g}{l}\,\lambda + \frac{g^2}{l}\,\frac{ml_1 + m_1 l_2}{A^0}\right\} = 0 \qquad (8.2)$$

One of the roots of the equation (8.2) is equal to zero, and the remaining four roots, can easily be shown to be located on the intervals

$$(-\infty, -\sqrt{g/l}),\ (-\sqrt{g/l}, 0)\ (0, \sqrt{g/l}),\ (\sqrt{g/l}, \infty)$$

Thus, the motion of the body possessing a spherical cavity completely filled with an ideal fluid, is always stable. It has the same character, as the motion of a dense rigid body undergoing a rotation, investigated in [2].

9. Let us now return to the study of the stability of the motion of a rigid body with

* With the reservation that the sign of the momentum K is opposite to that used in the case investigated in [1] and [3].

an ellipsoidal cavity completely filled with an ideal incompressible fluid. For that purpose we consider the characteristic equation (7.4) and attempt to determine the conditions for its roots to be real (i.e. the conditions of stability of motion).

A series of criteria are available for determining real roots of algebraic equations of the n-th order (See for instance [6]). However, in the present case (because of the complexity of the coefficients of the equations (7.4), these criteria are very cumbersome and consequently of little use for the study of stability conditions for the body under investigation, when the parameters entering the equation (7.1) of its motion, are subject to an arbitrary variation. Below is given a graphical-analytical method of investigation of the roots of the equation (7.4). It enables us to determine easily the zones of stable and unstable motion of the body in question for arbitrary values of its angular velocity ω.

In agreement with the rule of Descartes [7], the number of positive roots of the equation (7.4), for any values of the parameter $\omega > 0$, cannot be greater than three* and the number of negative roots cannot be greater than two. It will be shown later that the equation (7.4) for $\omega > 0$ has always two negative roots.

In fact, taking into consideration the equalities (7.2) and (7.5) we have

$$f(-\sqrt{g/l}, \omega) = \frac{g^2}{l^2} (A^* - A^\circ)(\omega\eta\sqrt{g/l}) > 0 \tag{9.1}$$

However, $f(-\infty; \omega) < 0$ and $f(0; \omega) < 0$. Consequently, in each of the intervals $-\infty < \lambda < -\sqrt{g/l}$ and $-\sqrt{g/l} < \lambda < 0$, there is one (negative) root of the equation (7.4).

To determine the character of the three remaining roots of the equation (7.4) let us collect the terms of its left hand side in decreasing powers of ω i.e.

$$p(\lambda)\,\omega^2 - q(\lambda)\omega + r(\lambda) = 0 \tag{9.2}$$

Where

$$p(\lambda) = \lambda\left(\lambda^2 - \frac{g}{l}\right)\varepsilon\eta^2, \quad q(\lambda) = \eta\left[R(\lambda) + \lambda^2\left(\lambda^2 - \frac{g}{l}\right)(\varepsilon - \varkappa)\right], \quad r(\lambda) = \lambda R(\lambda) \tag{9.3}$$

$$R(\lambda) = (A^0 + k\eta)\lambda^4 - \left[K + \frac{g}{l}(A^* + k\eta)\right]\lambda^2 + \frac{g}{l}K$$

$$\varepsilon = \frac{C\eta + k(1-\eta)}{\eta^2} > 0, \quad \varkappa = \frac{k(1-\eta)(1-\eta^2)}{\eta^2} > 0, \quad (\varepsilon - \varkappa) > 0 \tag{9.4}$$

Curves representing the polynomials $p(\lambda)$, $q(\lambda)$, and $r(\lambda)$ for $\lambda \geqslant 0$ are shown in Fig. 4. The analysis of the curves obtained shows that for $\lambda \geqslant 0$, the polynomial $r(\lambda)$ becomes equal to zero at $\lambda = 0$ and at two other values of λ denoted by λ_1 and λ_2. Similarly the polynomial $q(\lambda)$ is equal to zero only when $\lambda = \lambda_1^0$, and $\lambda = \lambda_2^0$. Finally, the polynomial $p(\lambda)$ is equal to zero either when $\lambda = 0$, or $\lambda = \sqrt{g}/\sqrt{l}$.

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* In case of $\omega = 0$, four roots of the equation (7.4) are real, and the fifth is equal to zero.

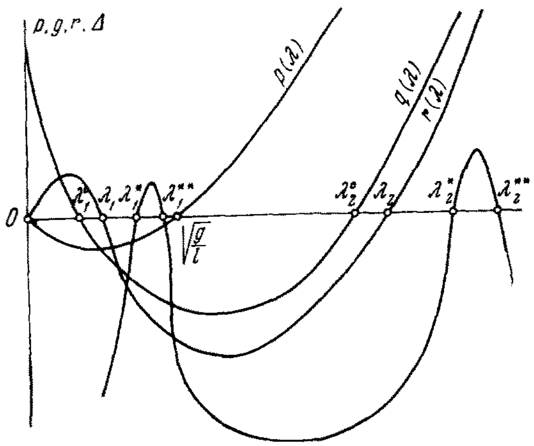

FIG. 4

It is possible to solve the equation (9.2) for ω and construct the curve representing the function

$$\omega(\lambda) = \frac{q(\lambda) \pm \sqrt{q^2(\lambda) - 4p(\lambda)r(\lambda)}}{2p(\lambda)} \tag{9.5}$$

From this curve, it is possible to establish for the given value of the angular velocity ω, whether all the roots of the equation (7.4) with respect to λ are real and, consequently, whether the motion of a rigid body with a fluid filled cavity is stable, or not.

Within the range of values of λ for which the discriminant

$$\Delta(\lambda) = 4p(\lambda)r(\lambda) - q^2(\lambda) \tag{9.6}$$

is positive, the values of ω, according to the formula (9.5) are complex conjugate quantities. This will occur in the intervals $(\lambda_1^*, \lambda_1^{**})$ and $(\lambda_2^*, \lambda_2^{**})$, where λ_1^*, λ_2^*, and λ_1^{**}, λ_2^{**} are the positive roots of the polynomial $\Delta(\lambda)$; these roots (as well as the negative roots possessing the same moduli) exist for any values of the parameters entering the expression (9.6). This can be proven by expressing the discriminant $\Delta(\lambda)$ in the form of a product

$$\Delta(\lambda) = -\eta^2 \left[R(\lambda) - \lambda^2\left(\lambda^2 - \frac{g}{l}\right)(\sqrt{\varepsilon} + \sqrt{\varkappa})^2\right] \times \\ \times \left[R(\lambda) - \lambda^2\left(\lambda^2 - \frac{g}{l}\right)(\sqrt{\varepsilon} - \sqrt{\varkappa})^2\right] \tag{9.7}$$

and taking into account the form of the polynomial $R(\lambda)$ according to the first formula (9.4).

According to the formula (9.5) both values of ω become equal for the values of λ, which coincide with one of the roots of the discriminant $\Delta(\lambda)$. Let us denote them by ω_1^*, ω_2^*, ω_1^{**} and ω_2^{**}. We find that they correspond to the cusps on the curve $\omega = \omega(\lambda)$. This curve is shown in Fig. 5. When constructing it, one must take into consideration the distribution of the zeros of the polynomials $p(\lambda)$, $q(\lambda)$ and $r(\lambda)$ determined by the formulas (9.3) and by the Fig. 4. In the intervals $(\omega_1^*, \omega_1^{**})$ and $(\omega_2^*, \omega_2^{**})$ (see Fig. 5), the fifth order equation (7.4) in λ has only three real roots (two of which are negative) for each calue of ω. The remaining two roots are complex.

Following the exposition made above, we come to the following conclusion: the motion of a rigid body with en ellipsoidal cavity completely filled with an ideal incompressible fluid is unstable for values of the angular velocity varying between the limits $\omega_1^* < \omega < \omega_1^{**}$ and $\omega_2^* < \omega < \omega_2^{**}$. Outside these intervals the motion of the body in question is stable.

10. The critical values $\omega_1^*, \omega_1^{**}, \omega_2^*$, and ω_2^{**} of the angular velocity are determined by the expression (9.5) in which we put

$$q^2(\lambda) - 4p(\lambda)r(\lambda) = -\Delta(\lambda) = 0$$

This results in the following simple formulas

$$\omega_1^* = \frac{q(\lambda_1^*)}{2p(\lambda_1^*)} = \frac{\varepsilon - \sqrt{\varepsilon\varkappa}}{\varepsilon\eta}\lambda_1^*, \qquad \omega_2^* = \frac{q(\lambda_2^*)}{2p(\lambda_2^*)} = \frac{\varepsilon - \sqrt{\varepsilon\varkappa}}{\varepsilon\eta}\lambda_2^*$$

$$\omega_1^{**} = \frac{q(\lambda_1^{**})}{2p(\lambda_1^{**})} = \frac{\varepsilon + \sqrt{\varepsilon\varkappa}}{\varepsilon\eta}\lambda_1^{**}, \qquad \omega_2^{**} = \frac{q(\lambda_2^{**})}{2p(\lambda_2^{**})} = \frac{\varepsilon + \sqrt{\varepsilon\varkappa}}{\varepsilon\eta}\lambda_2^{**} \qquad (10.1)$$

in which, as mentioned before λ_1^*, λ_1^{**}, λ_2^* and λ_2^{**} are respective positive roots of the discriminant $\Delta(\lambda)$.

11. As an example, we shall determine the critical values of the angular velocity ω of the rotation of a rigid body, the parameters of which are given below:

$$A = 15.84,\ C = 3.74\ [\text{gram sec}^2],\ m = 0.8145,\ m_1 = 0.0334\ [\text{gram sec}^2\ \text{cm}^{-1}]$$
$$a = 1.5,\ c = 4.35,\ l_1 = 6.6,\ l_2 = 5.5,\ l = 50\ [\text{cm}]$$

In this case, by the formulas (10.1) $\omega_1^* = 48$, $\omega_1^{**} = 53$, $\omega_2^* = 264$, and $\omega_2^{**} = 309$ [rev/min]. Thus, for values of ω $48 < \omega < 53$, and $264 < \omega < 309$ rev/min, the motion of the body in question is unstable. For all values of ω outside these intervals it is stable.

The critical values of the angular velocity were also calculated for the same parametric values by using the criterion given in [6]. They were found to be equal to those determined above, their determination however, required a much greater amount of work.

12. In order to check the criterion of stability obtained above for the motion of a rigid body with a fluid-filled cavity, experimental investigations described in [8], were made at the physico-technical laboratory of the Institute of Mechanics of the Academy of Sciences of the Ukrainian SSR by E.V. Virt, and A.P. Polyvianna under the supersions of Dr. Malashenko.

The model which consisted of a hollow body which had cylindrical insert inside it was fixed to the axle of a vertical motor by means of a thin string or a capron thread.

The insert consisted of two separate parts which could be hermetically sealed to produce an ellipsoidal cavity (see Fig.7 in [8]). In the top part of the insert there was an opening through which the cavity could be filled with fluid (in the experiments described, ethanol was used).

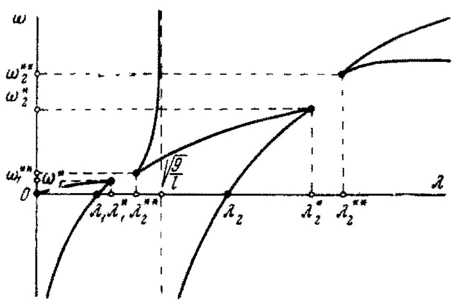

FIG. 5

The angular velocity of the model was varied between the limits of 100 and 3000 rev/min. The stabilization of the angular velocity was accomplished by means of a precision sound generator and a thyrathron.

The stability (or instability) of the motion of the model at various values of the angular velocity was determined by tapping the model slightly and observing the resulting behavior.

Experimental results with the model considered in section 11 have shown that the motion of the model was stable over the interval of 100-200 rev/min. In the 220-450 rev/min range it lost the stability but regained if above 450 rev/min.

The experimental determination of the limits of the first region of critical values of the angular velocity (see section 11 of this work) was not attempted. (The experimental set up could not be used for angular velocities less than 100 rev/min).

BIBLIOGRAPHY

1. Ishlinskii A. Iu., and Temchenko, M.E.O. Malykh kolebaniiakh vertikal'noi osi volchka, imeiushchego polost', tselikom napolnennuiu ideal'noi neszhimaemoi zhidkost'iu (On the small oscillations of the vertical axis of a gyroscope with a cavity completely filled with an ideal incompressible fluid). P.M.T.F., No. 3, 1960.

2. Morozova, E.P. Ob ustoichivosti vrashcheniia tverdogo tela, podveshennogo na strune (On the stability of rotation of a solid suspended on a string) *PMM*, Vol. 20, No. 5, 1956.

3. Sobolev, S.L. O dvizhenii simmetrichescogo volchka s polost'iu, napolnenoi zhidkost'iu. (On the motion of a gyroscope with a fluid filled cavity). P.M.T.F., No.3. 1960.

4. Suslov, G.K. Teoreticheskaia mekhanika (Theoretical mechanics) Gostekhizdat. 1946.

5. Kibel' N.E., Kochin, I.A., and Roze, N.B. Teoreticheskaia gidromekhanika (Theoretical fluid mechanics). Gostekhizdat, Vol. 1, 1948.

6. Chebotarev, N.G., and Meiman, N.N. Problema Raussa-Gurvittsa dlia polinomov i tselykh funktsii (The Routh-Hurwitz problem for polynomials and integral functions). Proceedings of the V.A. Steklov Mathematical Institute. Vol. 26, 1949. Edition of the Academy of Sciences of the USSR.

7. Kurosh A.G. Kurs vysshei algebry (Course in higher algebra) Gostekhizdat, 1955.

8. Malashenko S.V. and Temchenko, M.E. Ob odnom metode eksperimental'nogo issledovaniia ustoichivosti dvizheniia volchka, vnutri kotorogo imeetsia polost' napolnennaia zhidkost'iu (On an experimental method of investigation of the stability of motion of a gyroscope, inside which there is a fluid-filled cavity). P.M.T.F., No. 3, 1960.

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